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# Stationary patterns for a cooperative model with nonlinear diffusion

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## 1 Introduction

In this article we study positive steady-state solutions of the following strongly coupled reaction-diffusion system:

$$
\text{(P)}\begin{cases}
u_t = \Delta\left[\left(1+\dfrac{\alpha}{\mu+v}\right)u\right] + u(a-u+cv) & \text{in } \Omega\times(0,T),\\
v_t = \Delta v + v(-b+du-v) & \text{in } \Omega\times(0,T),\\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T),\\
u(\cdot,0)=u_0(\cdot),\quad v(\cdot,0)=v_0(\cdot) & \text{in } \Omega,
\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$; $a, b, c, d$ and $\mu$ are all positive constants; $\alpha$ is a non-negative constant; $u_0$ and $v_0$ are given non-negative functions which are not identically zero. System (P) is a Lotka-Volterra cooperative model with a density-dependent diffusion term of a fractional type; unknown functions $u$ and $v$ represent population densities of two cooperative species, respectively; $a$ and $-b$ denote the intrinsic growth rates of the respective species; $c$ and $d$ denote interaction coefficients. When $\alpha = 0$, (P) is reduced to a classical Lotka-Volterra cooperative model with diffusion. See [6] and [13] for such a cooperative model.

In the first equation of (P), the nonlinear diffusion term $\alpha\Delta\{u/(\mu+v)\}$ describes a situation where species $u$ tends to leave low-density areas of species $v$. This situation is natural because relations between $u$ and $v$ are cooperative. A population model with density-dependent diffusion was first proposed by Shigesada, Kawasaki and Teramoto [14] to investigate the habitat segregation phenomena between two competing species. Since their work, many mathematicians have studied population models with density-dependent diffusion. However, population models including density-dependent diffusion terms of a fractional type have appeared in recent years; for example, see [5], [16] for cooperative models with Dirichlet boundary conditions; [2], [3] for prey-predator models with Dirichlet boundary conditions; [12], [15] for three-species prey-predator models with Neumann boundary conditions. See also the monograph of Okubo and Levin [11] for the biological background.

The stationary problem associated with (P) is

$$\text{(SP)}\begin{cases} \Delta\left[\left(1+\dfrac{\alpha}{\mu+v}\right)u\right]+u(a-u+cv)=0 & \text{in } \Omega, \\ \Delta v+v(-b+du-v)=0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0 & \text{on } \partial\Omega. \end{cases}$$

Our main purpose is to study the existence of stationary patterns (i.e. positive non-constant solutions) for (SP) with the weak cooperative condition

$$\frac{a}{b}>\frac{1}{d}>c. \tag{1.1}$$

From now on, we will always assume (1.1). It is well known that, if $\alpha=0$, then every solution of (P) converges to a unique positive constant steady-state

$$(u^*,v^*):=\left(\frac{a-bc}{1-cd},\frac{ad-b}{1-cd}\right)$$

uniformly as $t\to\infty$; see [6]. This implies the following proposition.

**Proposition 1.1.** *Let $\alpha=0$. Then $(u^*,v^*)$ is a unique positive solution of* (SP).

Proposition 1.1 means that no stationary pattern exists in the linear diffusion case. However, the presence of density-dependent diffusion enables us to construct stationary patterns of (SP). We focus on $\alpha$ to show the emergence of stationary patterns for (SP).

Let $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ denote eigenvalues of $-\Delta$ with the homogeneous Neumann boundary condition on $\partial\Omega$ and let $m_i$ denote the algebraic multiplicity of $\lambda_i$. Then we have the following theorem.

**Theorem 1.1.** *Suppose that $\{v^*(b-\mu)\}/(\mu+v^*)\in(\lambda_l,\lambda_{l+1})$ for some $l\geq 1$ and that $\sum_{i=1}^{l} m_i$ is odd. Then there exists a positive constant $\alpha^*=\alpha^*(a,b,c,d,\mu)$ such that* (SP) *has at least one positive non-constant solution for each $\alpha>\alpha^*$.*

We are also interested in the limiting patterns of (SP) as $\alpha\to\infty$. Under the restriction $N\leq 3$, we obtain the following limiting system as $\alpha\to\infty$.

**Theorem 1.2.** *Suppose $N \le 3$ and $b > \mu$. Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ be any sequence such that $\lim_{i\to\infty} \alpha_i = \infty$ and positive functions $(u_i, v_i)$ satisfy* (SP) *with $\alpha = \alpha_i$. Then, by passing to a subsequence if necessary, it holds that*

$$\lim_{i\to\infty}(u_i, v_i) = (\tau(\mu + \bar{v}), \bar{v}) \quad \text{in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}),$$

*where $\tau$ is a positive constant satisfying $1 < d\tau < b/\mu$, $\bar{v}$ is a positive function in $\Omega$ and $(\tau, \bar{v})$ satisfies*

$$\begin{cases} \Delta\bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 & \text{in } \Omega, \\ \dfrac{\partial\bar{v}}{\partial n} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega (\mu + \bar{v})\{a - \tau\mu + (c - \tau)\bar{v}\}dx = 0. \end{cases} \tag{1.2}$$

We expect that the limiting system (1.2) may give much information on profiles of stationary patterns of (SP) for large $\alpha$. We will give some remarks about (1.2) in the last section.

Throughout the article, the usual norms of $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\bar{\Omega})$ are defined by

$$\|\psi\|_p := \left(\int_\Omega |\psi(x)|^p dx\right)^{1/p} \quad \text{and} \quad \|\psi\|_\infty := \max_{x\in\bar{\Omega}} |\psi(x)|,$$

respectively.

## 2 Stability of the constant solution $(u^*, v^*)$

In this section, we will analyze the linearized stability of the constant stationary solution $(u^*, v^*)$ for (P).

The linearized eigenvalue problem of (P) at $(u^*, v^*)$ is given by

$$\begin{cases} -\left(1 + \dfrac{\alpha}{\mu + v^*}\right)\Delta h + \dfrac{\alpha u^*}{(\mu + v^*)^2}\Delta k + u^* h - c u^* k = \eta h & \text{in } \Omega, \\ -\Delta k - d v^* h + v^* k = \eta k & \text{in } \Omega, \\ \dfrac{\partial h}{\partial n} = \dfrac{\partial k}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We know that $(u^*, v^*)$ is linearly stable when $\alpha = 0$. Using the expansions of $h$ and $k$ in terms of eigenfunctions of $-\Delta$, one can see that $\eta$ is an eigenvalue of (2.1) if and only if

$$\det\begin{pmatrix} -\eta + \left(1 + \frac{\alpha}{\mu+v^*}\right)\lambda_i + u^* & -\frac{\alpha u^*}{(\mu+v^*)^2}\lambda_i - c u^* \\ -d v^* & -\eta + \lambda_i + v^* \end{pmatrix} = 0$$

for some $i \geq 0$. In particular, $\eta = 0$ is an eigenvalue of (2.1) if and only if

$$\frac{\lambda_i}{(\mu+v^*)^2}\{(\mu+v^*)(\lambda_i+v^*) - du^*v^*\}\alpha + (\lambda_i+u^*)(\lambda_i+v^*) - cdu^*v^* = 0$$

for some $i \geq 0$. Note that $(\lambda_i+u^*)(\lambda_i+v^*) - cdu^*v^* > 0$ for all $i \geq 0$ because of (1.1). Thus it is easy to see that the linearized stability of $(u^*, v^*)$ changes as $\alpha$ increases in (P) if and only if

$$\begin{aligned}(\mu+v^*)(\lambda_1+v^*) - du^*v^* &= (\mu+v^*)\lambda_1 + v^*(\mu+v^*-du^*)\\ &= (\mu+v^*)\lambda_1 + v^*(\mu-b)\\ &< 0.\end{aligned}$$

Therefore, $b > \mu$ is necessary for the linearized stability of $(u^*, v^*)$ to change (and so we do not discuss the case $b \leq \mu$, especially, $-b \geq 0$). This means that the difference in the intrinsic growth rates between two species $u$ and $v$ contributes to creating stationary patterns in (SP).

# 3 Proof of Theorem 1.1

## 3.1 Reduction to the semilinear system

Our method of the proof of Theorem 1.1 will be based on the Leray-Schauder degree theory (see e.g., [9]) and some a priori estimates. We first introduce a new unknown function $U$ by

$$U = \left(1 + \frac{\alpha}{\mu+v}\right)u. \tag{3.1}$$

Clearly, there exists a one-to-one correspondence between $(u, v) > 0$ and $(U, v) > 0$. As far as we discuss positive solutions, (SP) is rewritten in the following equivalent form:

$$\text{(EP)}\begin{cases}\Delta U + \dfrac{\mu+v}{\mu+v+\alpha}U\left(a - \dfrac{\mu+v}{\mu+v+\alpha}U + cv\right) = 0 & \text{in } \Omega,\\ \Delta v + v\left(-b + d\dfrac{\mu+v}{\mu+v+\alpha}U - v\right) = 0 & \text{in } \Omega,\\ \dfrac{\partial U}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega.\end{cases}$$

Thus, it is sufficient to show the existence of positive non-constant solutions of (EP).

## 3.2 A priori estimates

In this subsection, we will give some a priori estimates for positive solutions of (EP). Before stating the a priori estimates, we recall the following maximum principle due to Lou and Ni [7].

**Lemma 3.1.** *Suppose that $g \in C(\bar{\Omega} \times \mathbb{R})$.*
(i) *If $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \quad in \ \Omega, \quad \frac{\partial w}{\partial n} \leq 0 \quad on \ \partial\Omega,$$

*and $w(x_0) = \max_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$.*
(ii) *If $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \quad in \ \Omega, \quad \frac{\partial w}{\partial n} \geq 0 \quad on \ \partial\Omega,$$

*and $w(x_0) = \min_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.*

Now we can derive the following a priori estimates.

**Lemma 3.2.** *Let $\zeta$ be any fixed positive number. Then there exist two positive constants $C_*(\zeta) = C_*(\zeta, a, b, c, d, \mu) < C^*(\zeta) = C^*(\zeta, a, b, c, d, \mu)$ such that, if $\alpha \leq \zeta$, then any positive solution $(U, v)$ of (EP) satisfies*

$$a \leq U(x) \leq C^*(\zeta) \quad and \quad C_*(\zeta) \leq v(x) \leq C^*(\zeta) \quad for\ all \ x \in \bar{\Omega}.$$

*Proof.* Let $U(x_0) = \max_{\bar{\Omega}} U$ and $v(y_0) = \max_{\bar{\Omega}} v$ with $x_0, y_0 \in \bar{\Omega}$. Applying Lemma 3.1 to (EP), we have

$$\max_{\bar{\Omega}} U \leq \frac{\mu + v(x_0) + \alpha}{\mu + v(x_0)}(a + cv(x_0))$$

and

$$\max_{\bar{\Omega}} v \leq -b + d\frac{\mu + v(y_0)}{\mu + v(y_0) + \alpha}U(y_0) \leq -b + d\max_{\bar{\Omega}} U. \tag{3.2}$$

Thus

$$\begin{aligned}\max_{\bar{\Omega}} U &\leq a + cv(x_0) + \zeta\frac{a + cv(x_0)}{\mu + v(x_0)} \\ &\leq a + c(-b + d\max_{\bar{\Omega}} U) + \zeta \max\left\{\frac{a}{\mu}, c\right\}.\end{aligned}$$

Therefore, we see

$$\max_{\bar{\Omega}} U \leq \frac{a - bc + \zeta \max\{a/\mu, c\}}{1 - cd}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\max_{\bar{\Omega}} v \le -b + \frac{d(a - bc + \zeta \max\{a/\mu, c\})}{1 - cd} = \frac{ad - b + \zeta d \max\{a/\mu, c\}}{1 - cd}. \tag{3.4}$$

Hence we have obtained the desired upper bound of $(U, v)$.

Let $U(z_0) = \min_{\bar{\Omega}} U$ with some $z_0 \in \bar{\Omega}$. Using Lemma 3.1 to the first equation of (EP), we get

$$\min_{\bar{\Omega}} U \ge \frac{\mu + v(z_0) + \alpha}{\mu + v(z_0)}(a + cv(z_0)) \ge a. \tag{3.5}$$

Thus we have obtained the desired lower bound of $U$.

Finally, we derive a lower bound of $v$ by contradiction. Suppose that there exist a certain positive constant $\zeta_0$ and a sequence $\{(U_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ such that $\alpha_i \le \zeta_0$ for all $i \in \mathbb{N}$, $\lim_{i\to\infty} \alpha_i = \alpha_\infty$ for some non-negative constant $\alpha_\infty$,

$$\lim_{i\to\infty} \min_{\bar{\Omega}} v_i = 0 \tag{3.6}$$

and positive functions $(U_i, v_i)$ satisfy

$$\begin{cases} \Delta U_i + \dfrac{\mu + v_i}{\mu + v_i + \alpha_i} U_i \left( a - \dfrac{\mu + v_i}{\mu + v_i + \alpha_i} U_i + cv_i \right) = 0 & \text{in } \Omega, \\ \Delta v_i + v_i \left( -b + d \dfrac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) = 0 & \text{in } \Omega, \\ \dfrac{\partial U_i}{\partial n} = \dfrac{\partial v_i}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

By using the regularity theory for elliptic equations (see e.g., [1]) to the second equation of (3.7), it follows from (3.3) and (3.4) that

$$\|v_i\|_{W^{2,p}(\Omega)} \le C(\zeta_0)$$

with some positive constant $C(\zeta_0) = C(\zeta_0, a, b, c, d, \mu)$ independent of $i$. If $p > N$, then Sobolev's embedding theorem implies $\{v_i\}_{i=1}^{\infty}$ is compact in $C^1(\bar{\Omega})$. Consequently, there exists a subsequence, which is still denoted by $\{v_i\}_{i=1}^{\infty}$, such that

$$\lim_{i\to\infty} v_i = v_\infty \quad \text{in } C^1(\bar{\Omega}) \tag{3.8}$$

with some non-negative function $v_\infty \in C^1(\bar{\Omega})$. Similarly, there exists a non-negative function $U_\infty \in C^1(\bar{\Omega})$ such that

$$\lim_{i\to\infty} U_i = U_\infty \quad \text{in } C^1(\bar{\Omega}). \tag{3.9}$$

Therefore, $v_\infty$ satisfies

$$\Delta v_\infty + v_\infty \left( -b + d \frac{\mu + v_\infty}{\mu + v_\infty + \alpha_\infty} U_\infty - v_\infty \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial v_\infty}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in a weak sense. By standard elliptic regularity theory we have $v_\infty \in C^2(\bar{\Omega})$ and thus $v_\infty$ is a classical solution of the above equation. Then it follows from (3.6),(3.8) and the strong maximum principle that $v_\infty \equiv 0$ in $\bar{\Omega}$. We can easily see from the above argument that $U_\infty$ satisfies

$$\Delta U_\infty + \frac{\mu}{\mu+\alpha_\infty} U_\infty \left(a - \frac{\mu}{\mu+\alpha_\infty} U_\infty\right) = 0 \quad \text{in } \Omega, \quad \frac{\partial U_\infty}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in the classical sense. Then by the strong maximum principle and Lemma 3.1, either $U_\infty \equiv a(\mu+\alpha_\infty)/\mu$ or $U_\infty \equiv 0$ in $\bar{\Omega}$. Combining (3.5) and (3.9), we can conclude $U_\infty \equiv a(\mu+\alpha_\infty)/\mu$ in $\bar{\Omega}$. Hence

$$\lim_{i\to\infty} \left(-b + d\frac{\mu+v_i}{\mu+v_i+\alpha_i} U_i - v_i\right) = ad - b > 0 \quad \text{uniformly in } \Omega$$

by (1.1) and this means

$$v_i \left(-b + d\frac{\mu+v_i}{\mu+v_i+\alpha_i} U_i - v_i\right) > 0 \quad \text{in } \Omega$$

for sufficiently large $i \in \mathbb{N}$ because $v_i > 0$ in $\Omega$. On the other hand, from the second equation of (3.7), we have

$$\int_\Omega v_i \left(-b + d\frac{\mu+v_i}{\mu+v_i+\alpha_i} U_i - v_i\right) dx = -\int_\Omega \Delta v_i dx = -\int_{\partial\Omega} \frac{\partial v_i}{\partial n} d\sigma = 0$$

for all $i \in \mathbb{N}$. This is a contradiction; thus our proof is complete. □

## 3.3 Completion of the proof of Theorem 1.1

Set $X = C(\bar{\Omega}) \times C(\bar{\Omega})$. For each $\alpha \geq 0$, define an operator $F_\alpha$ by

$$F_\alpha \begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} (-\Delta + I)^{-1} \left[U + \frac{\mu+v}{\mu+v+\alpha} U \left(a - \frac{\mu+v}{\mu+v+\alpha} U + cv\right)\right] \\ (-\Delta + I)^{-1} \left[v + v\left(-b + d\frac{\mu+v}{\mu+v+\alpha} U - v\right)\right] \end{pmatrix},$$

where $I$ is the identity map from $C(\bar{\Omega})$ into itself, and $(-\Delta + I)^{-1}$ is the inverse operator of $-\Delta + I$ subject to the homogeneous Neumann boundary condition on $\partial\Omega$. It is easy to see that $F_\alpha : X \to X$ is well-defined, and that by elliptic regularity theory and Sobolev's embedding theorem, $F_\alpha$ is a continuous and compact operator for each $\alpha \geq 0$. From these observations, one can define the Leray-Schauder degree of $I - F_\alpha$ at 0 in a suitable open set. Furthermore, $(U, v)$ is a positive solution of $(I - F_\alpha)(U, v) = 0$ if and only if $(U, v)$ is a positive solution of (EP).

In view of (3.1), we set

$$U_\alpha^* = \left(1 + \frac{\alpha}{\mu + v^*}\right) u^*.$$

Hence $(U_\alpha^*, v^*)$ is a zero point of $I - F_\alpha$. Then we can calculate the index of $I - F_0$ at $(u^*, v^*)$ and the index of $I - F_\alpha$ at $(U_\alpha^*, v^*)$ for sufficiently large $\alpha$, which are denoted by $\mathrm{index}(I - F_0, (u^*, v^*))$ and $\mathrm{index}(I - F_\alpha, (U_\alpha^*, v^*))$, respectively. We refer to [10] for the proofs of Lemmas 3.3 and 3.4.

**Lemma 3.3.** *It holds that* $\mathrm{index}(I - F_0, (u^*, v^*)) = 1$.

**Lemma 3.4.** *Suppose that* $\{v^*(b-\mu)\}/(\mu + v^*) \in (\lambda_l, \lambda_{l+1})$ *for some* $l \geq 1$. *Then there exists a positive constant* $\alpha^* = \alpha^*(a, b, c, d, \mu)$ *such that, if* $\alpha > \alpha^*$, *then*

$$\mathrm{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^{l} m_i},$$

*where* $m_i$ *denotes the algebraic multiplicity of* $\lambda_i$ *defined in Section 1.*

By virtue of Lemmas 3.3 and 3.4, we are ready to prove Theorem 1.1. In the proof of Theorem 1.1, we represent (EP) as $(\mathrm{EP})_\alpha$ to indicate the dependence on $\alpha$.

*Proof of Theorem 1.1.* Fix any $\alpha > \alpha^*$, where $\alpha^*$ is a constant given in Lemma 3.4. It follows from Lemma 3.2 that there exist two positive constants $C_*(\alpha) = C_*(\alpha, a, b, c, d, \mu) < C^*(\alpha) = C^*(\alpha, a, b, c, d, \mu)$ such that

$$a \leq U(x) \leq C^*(\alpha) \quad \text{and} \quad C_*(\alpha) \leq v(x) \leq C^*(\alpha) \quad \text{for all } x \in \bar{\Omega}$$

for any positive solution $(U, v)$ of $(\mathrm{EP})_\nu$ with any $\nu \in [0, \alpha]$. We define

$$S = \left\{ (U, v) \in X \mid \frac{a}{2} \leq U \leq 2C^*(\alpha),\ \ \frac{C_*(\alpha)}{2} \leq v \leq 2C^*(\alpha) \ \text{ in } \bar{\Omega} \right\};$$

so that $I - F_\nu$ has no zero point on the boundary of $S$ for any $\nu \in [0, \alpha]$. Note that $I - F_0$ has a unique zero point $(u^*, v^*)$ in $S$. On account of the homotopy invariance of the Leray-Schauder degree and Lemma 3.3, we have

$$\deg(I - F_\alpha, S, 0) = \deg(I - F_0, S, 0) = \mathrm{index}(I - F_0, (u^*, v^*)) = 1. \tag{3.10}$$

Suppose that $(\mathrm{EP})_\alpha$ has no positive non-constant solution, i.e. $I - F_\alpha$ has a unique zero point $(U_\alpha^*, v^*)$ in $S$. Then from the assumption $\sum_{i=1}^{l} m_i$ being odd and Lemma 3.4, it follows that

$$\deg(I - F_\alpha, S, 0) = \mathrm{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^{l} m_i} = -1,$$

which contradicts (3.10). Thus we complete the proof. □

# 4 Proof of Theorem 1.2

We first state some a priori estimates independent of $\alpha$.

**Lemma 4.1.** *Suppose that $N \leq 3$. Then there exists a positive constant $C_0 = C_0(a, b, c, d, \mu)$ independent of $\alpha$ such that any positive solution $(u, v)$ of* (SP) *satisfies*

$$\|u\|_\infty \leq C_0 \quad \textit{and} \quad \|v\|_\infty \leq C_0.$$

Lemma 4.1 can be proved by combining the $L^2$-estimates for positive solutions of (SP) (independent of $\alpha$ and $N$) with Harnack inequality (due to Lin, Ni and Takagi [4], and Lou and Ni [8]). We refer to [10] for the proof of Lemma 4.1.

We are now in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^\infty$ be any sequence such that $\lim_{i\to\infty} \alpha_i = \infty$ and positive functions $(u_i, v_i)$ satisfy (SP) with $\alpha = \alpha_i$. Set

$$\psi_i = \left(\frac{1}{\alpha_i} + \frac{1}{\mu + v_i}\right) u_i.$$

Note that positive functions $(\psi_i, v_i)$ satisfy

$$\begin{cases} \Delta\psi_i + \dfrac{u_i(a - u_i + cv_i)}{\alpha_i} = 0 & \text{in } \Omega, \\ \Delta v_i + v_i(-b + du_i - v_i) = 0 & \text{in } \Omega, \\ \dfrac{\partial \psi_i}{\partial n} = \dfrac{\partial v_i}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and that $\{\psi_i\}_{i=1}^\infty$ is bounded independently of $i$ by Lemma 4.1. Then by the compactness argument as in the proof of (3.8), there exists a subsequence, which is still denoted by $\{\psi_i\}_{i=1}^\infty$, such that

$$\lim_{i\to\infty} \psi_i = \tau \quad \text{in } C^1(\bar{\Omega})$$

for a non-negative function $\tau \in C^1(\bar{\Omega})$. Similarly, we see

$$\lim_{i\to\infty} v_i = \bar{v} \quad \text{in } C^1(\bar{\Omega}) \tag{4.1}$$

for a non-negative function $\bar{v} \in C^1(\bar{\Omega})$. Therefore, we obtain

$$\lim_{i\to\infty} u_i = \lim_{i\to\infty} \frac{\psi_i}{1/\alpha_i + 1/(\mu + v_i)} = \tau(\mu + \bar{v}) \quad \text{in } C^1(\bar{\Omega}). \tag{4.2}$$

We will show that $\tau$ is a positive constant. Observe that $\tau$ satisfies

$$\Delta\tau = 0 \quad \text{in } \Omega, \quad \frac{\partial\tau}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in a weak sense. A standard elliptic regularity theory ensures $\tau \in C^2(\bar{\Omega})$; so that $\tau$ must be a non-negative constant. Let $v_i(x_i) = \max_{\bar{\Omega}} v_i$ with some $x_i \in \bar{\Omega}$. It follows from Lemma 3.1 that

$$u_i(x_i) \geq \frac{b + v_i(x_i)}{d} > \frac{b}{d}\ (> 0)$$

for all $i \in \mathbb{N}$. This fact, together with (4.2), yields $\tau > 0$.

We next prove $(\tau, \bar{v})$ satisfies (1.2). Note that $\bar{v}$ satisfies

$$\Delta\bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 \quad \text{in } \Omega, \quad \frac{\partial\bar{v}}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{4.3}$$

in a weak sense. In the standard manner, one can see that $\bar{v} \in C^2(\bar{\Omega})$ and $\bar{v}$ is a classical nonnegative solution of (4.3). It follows from the strong maximum principle that either $\bar{v} \equiv 0$ or $\bar{v} > 0$ in $\Omega$. We show $\bar{v} > 0$ in $\Omega$ by contradiction. Suppose that $\bar{v} \equiv 0$ in $\Omega$. Then it follows from (4.1) and (4.2) that

$$\lim_{i\to\infty} a - u_i + cv_i = a - \tau\mu \quad \text{and} \quad \lim_{i\to\infty} -b + du_i - v_i = -b + d\tau\mu$$

uniformly in $\Omega$. On the other hand,

$$\int_\Omega u_i(a - u_i + cv_i)dx = \int_\Omega v_i(-b + du_i - v_i)dx = 0 \tag{4.4}$$

for all $i \in \mathbb{N}$. Consequently, $a - \tau\mu = -b + d\tau\mu = 0$ because of $u_i > 0$ and $v_i > 0$ in $\Omega$ and thus $ad - b = 0$. This contradicts (1.1). Therefore $\bar{v} > 0$ in $\Omega$.

By (4.1), (4.2) and (4.4), it is clear that

$$\int_\Omega (\mu + \bar{v})\{a - \tau\mu + (c - \tau)\bar{v}\}dx = \int_\Omega (\mu + \bar{v})\{a - \tau(\mu + \bar{v}) + c\bar{v}\}dx = 0.$$

Hence it only remains to show $1 < d\tau < b/\mu$. By the assumption of Theorem 1.2,

$$-b + d\tau\mu < -\mu + d\tau\mu = \mu(d\tau - 1).$$

It thus follows from Lemma 3.1 and (4.3) that if $d\tau - 1 \leq 0$, then $\max_{\bar{\Omega}} \bar{v} \leq 0$ and this contradicts $\bar{v} > 0$ in $\Omega$. Therefore, $d\tau > 1$. Using Lemma 3.1 and $\bar{v} > 0$ in $\Omega$ again, we obtain $d\tau < b/\mu$. Hence we complete the proof. □

# 5 Remarks about the limiting system (1.2)

We easily see that $(\tau, \bar{v}) = (u^*/(\mu+v^*), v^*)$ is the only positive constant solution of (1.2). So our concern is about positive non-constant solutions of (1.2). We discuss the differential equations without the integral constraint in (1.2) under the restriction $N \leq 3$:

$$\begin{cases} \Delta\bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 & \text{in } \Omega, \\ \dfrac{\partial\bar{v}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Set

$$w = \frac{d\tau - 1}{b - d\tau\mu}\bar{v},$$

where $1 < d\tau < b/\mu$. Then (5.1) is rewritten in the following equivalent form:

$$\begin{cases} \dfrac{1}{b - d\tau\mu}\Delta w - w + w^2 = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

We note that, if $(0 <)b - d\tau\mu \ll 1$, then (5.2) has no positive non-constant solution (see [4]). Therefore, $b \gg 1$ is necessary for (1.2) to have positive non-constant solutions. We will study (1.2) in detail in the future.

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